Interpretation of Black Body Radiation as a Decay Process

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Abstract

The treatment of black body radiation as a decay process with the wavelength \( \lambda \) as the time marker, leads to a function \( D_\lambda \) that apportions the total thermodynamic Stefan-Boltzmann emitted intensity \( I \) over the entire wavelength range. The resulting distribution function

\[
I_\lambda = ID_\lambda = \frac{T^6}{b^2} \lambda e^{-\frac{T \lambda}{b}}
\]

gives the Stefan-Boltzmann law on integration over the same interval. Differentiation of \( I_\lambda \) produces Wien’s displacement law as the condition for the wavelength \( \lambda_m \) at maximum emitted intensity. Substitution of \( \lambda_m \) in \( I_\lambda \) yields the maximum emitted intensity \( I_{\lambda_m} \) as being proportional to \( T^5 \).

Hence \( I_\lambda \) satisfies exactly the three known empirical laws of black body radiation and fulfils Einstein’s hope for a solution of the radiation problem without the use of light quanta. In addition, the replacement of \( \frac{T}{b} \) with a single constant \( G \) simplifies the distribution function so that

\[
I_\lambda = \sigma_0 G^6 \lambda e^{-G \lambda}
\]

where \( \sigma_0 = b^4 \sigma \). Consequently \( G \) defines a new temperature scale with units of reciprocal wavelength that unifies the thermodynamic and colour scales.

Key Words: Black body, decay process, light quanta, energy - mass principle, thermodynamic temperature, colour temperature.
**Introduction**

The total emitted intensity \( I \) of a black body is given by the Stefan-Boltzmann Law, which can be derived from classical thermodynamics [1]:

\[
I = \sigma T^4
\]

where \( T \) is the absolute temperature and \( \sigma \) is the Stefan-Boltzmann constant. The ideal solution for the emitted intensity as a function of wavelength \( (I_{\lambda}) \) or frequency \( (I_\nu) \) would be a product of the previous function \( (I) \) and a function \( (D_\lambda) \) or \( (D_\nu) \) that apportions the amount of energy given off at each \( \lambda \) or \( \nu \):

\[
I_{\lambda} d\lambda = ID_\lambda d\lambda = \sigma T^4 D_\lambda d\lambda \quad \text{2a)}
\]

\[
I_\nu d\nu = ID_\nu d\nu = \sigma T^4 D_\nu d\nu \quad \text{2b)}
\]

The idea to be used in deriving this apportioning function is that the energy emitted by a black body is proportional to its mass and is also a function of its absolute temperature and the time available for emission. Since the wavelength \( (\lambda) \) can be directly used as a time marker (it is directly proportional to time whereas the frequency \( (\nu) \) is inversely proportional), the derivation is easier with the wavelength. The reasoning behind this analysis is based on the energy – mass relationship derived from Einstein’s special theory of relativity [2]:

\[
E = mc^2 \quad \text{3)}
\]

**Derivation**

At any temperature \( (T) \), the mass loss at a given wavelength \( (dm_{\lambda}) \) can be treated as a typical decay process[3] with \( \lambda \) as the time marker:

\[
\frac{dm_{\lambda}}{d\lambda} = -Gm_{\lambda} = -\frac{T}{b}m_{\lambda} \quad \text{4)}
\]
where the proportionality constant $G$ has been written as the ratio $\frac{T}{b}$. Then:

$$\int_{m_0}^{m_\lambda} \frac{dm_{\lambda}}{m_{\lambda}} = -\frac{T}{b} \int_0^{\lambda} d\lambda$$  \hspace{1cm} (5)$$

where $m_0$ is the mass at the start of the decay process when $t = 0$, $(\lambda = 0)$.

Integration gives:

$$\ln \frac{m_{\lambda}}{m_0} = -\frac{T}{b} \lambda \; ; \; \; m_{\lambda} = m_0 e^{-\frac{T}{b} \lambda}$$  \hspace{1cm} (6)$$

For a small increment in time $(d\lambda)$ from $\lambda \to \lambda + d\lambda$ there is a small decrease in mass $(dm_{\lambda})$ from $m_{\lambda} \to m_{\lambda} - dm_{\lambda}$, so that equation 5) becomes:

$$\int_{m_\lambda}^{m_\lambda - dm_{\lambda}} \frac{dm_{\lambda}}{m_{\lambda}} = -\frac{T}{b} \int_{\lambda}^{\lambda + d\lambda} d\lambda$$  \hspace{1cm} (7)$$

Integration gives:

$$\ln \frac{m_\lambda - dm_{\lambda}}{m_\lambda} = -\frac{T}{b} (\lambda + d\lambda - \lambda) = -\frac{T}{b} d\lambda$$  \hspace{1cm} (8a)$$

$$\frac{m_\lambda - dm_{\lambda}}{m_\lambda} = e^{-\frac{T}{b} d\lambda}$$  \hspace{1cm} (8b)$$

$$m_\lambda - dm_{\lambda} = m_\lambda e^{-\frac{T}{b} d\lambda}$$  \hspace{1cm} (8c)$$

$$dm_{\lambda} = m_\lambda \left(1 - e^{-\frac{T}{b} d\lambda}\right)$$  \hspace{1cm} (8d)$$

On expansion this gives:

$$dm_{\lambda} = m_\lambda \left(1 - 1 + \frac{T}{b} d\lambda\right) = m_\lambda \frac{T}{b} d\lambda$$  \hspace{1cm} (9)$$

Substituting equation 6) in equation 9) gives:
\[ dm_\lambda = \frac{T}{b} m_0 e^{-\frac{T}{b} \lambda} d\lambda \]  

10) 

According to Einstein’s energy - mass relationship (equation 3)), this mass loss is accompanied by the emission of energy given by:

\[ dE_\lambda = (dm_\lambda) c^2 = \frac{T}{b} m_0 c^2 e^{-\frac{T}{b} \lambda} d\lambda \]  

11) 

Equation 11) defines the rate of emission of energy; \( \lambda \) of course being the time marker. The detection of energy at a given wavelength requires a sampling time corresponding to that wavelength. The apportioning function is thus considered to be of the form of equation 11) multiplied by that wavelength. Hence:

\[ D_\lambda d\lambda = \lambda dE_\lambda = m_0 c^2 \frac{T}{b} \lambda e^{-\frac{T}{b} \lambda} d\lambda \]  

12) 

This function can now be normalized by integration over the entire wavelength range. The resulting normalized function will then apportion the whole of the total Stefan - Boltzmann emitted intensity over this range. Thus:

\[ \int_0^\infty \lambda e^{-\frac{T}{b} \lambda} d\lambda = \frac{b}{T} \left[ \int e^{-\frac{T}{b} \lambda} d\lambda - \lambda e^{-\frac{T}{b} \lambda} \right]_0^\infty \]  

13a) 

\[ = -\frac{b}{T} \left[ \frac{be^{-\frac{T}{b} \lambda}}{T} + \lambda e^{-\frac{T}{b} \lambda} \right]_0^\infty \]  

13b) 

\[ = -\frac{b}{T} \left[ 0 + 0 - \frac{b}{T} - 0 \right] = \frac{b^2}{T^2} \]  

13c) 

\[ \int_0^\infty D_\lambda d\lambda = m_0 c^2 \frac{T}{b} \frac{b^2}{T^2} = \frac{m_0 c^2 b}{T} \]  

14) 

The normalized apportioning function \( D_\lambda \) can thus be defined as:

\[ D_\lambda = \frac{T}{m_0 c^2 b} D_\lambda = \frac{T^2}{b^2} \lambda e^{-\frac{T}{b} \lambda} \]  

15)
The black body radiation law becomes:

\[ I_\lambda d\lambda = I_0 \lambda d\lambda = \left[ \sigma T^4 \left( \frac{T^2}{b^2} \lambda e^{-\frac{T}{b\lambda}} \right) \right] d\lambda = \frac{\sigma T^6}{b^2} \lambda e^{-\frac{T}{b\lambda}} d\lambda \quad (16) \]

### Analysis

It is now necessary to analyse this function and compare its behaviour to known experimental data [4,5]. The wavelength \( \lambda_m \) at maximum emitted intensity is found by equating the derivative to zero

\[ \frac{dI_\lambda}{d\lambda} = \frac{\sigma T^6}{b^2} e^{-\frac{T}{b\lambda}} \left[ 1 - \frac{T}{b\lambda} \right] = 0 \quad (17) \]

This implies that:

\[ \lambda_m T = b = 2.90 \times 10^{-3} \text{ mK} \quad (18) \]

This analysis thus directly yields Wien’s displacement law [4,5,6] as the requirement for the wavelength \( \lambda_m \) at the maximum emitted intensity since the derived constant \( b \) must be identical to Wien’s displacement constant. Substituting \( \lambda_m \) in equation 16) gives:

\[ I_{\lambda_m} = \frac{\sigma T^6}{b^2} \frac{b}{T} e^{-1} = \frac{\sigma}{b e} T^5 \quad (19) \]

which agrees with the observation that the maximum emitted intensity is directly proportional to \( T^5 \). Because of the form of equation 16), its integration over the entire wavelength range gives the total emitted intensity to be identical to the Stefan-Boltzmann law of equation 1). The great advantage of this approach is its direct relationship with the result derived by Boltzmann on the basis of classical thermodynamics [1]. This law (equation 16)) thus satisfies exactly the three known empirical requirements (equations 18), 19) and 1)) of black body radiation [4].
This derivation also suggests that there is really no reason for using the proportionality constant \((G)\) in equation 4) as a ratio of two constants \((\frac{T}{b})\). This is solely due to the arbitrarily chosen size of the unit of temperature. Reverting to a single constant by substituting in equation 16) gives:

\[
I_\lambda d\lambda = \frac{\sigma T^6}{b^2} \lambda e^{-\frac{T}{b} \lambda} d\lambda = \sigma G^2 b^4 G^4 \lambda e^{-G \lambda} d\lambda = \sigma_G^6 \lambda e^{G \lambda} d\lambda
\]

where \(\sigma_G = b^4 \sigma = 3.998 \times 10^{-18} \text{ Wm}^2\). This result thus suggests that the proportionality constant \((G)\) defines a new temperature scale with units of \(\frac{1}{\text{Wavelength}}\); \(G = \left(\frac{\circ \text{K}}{2.90 \times 10^{-3}}\right) \text{ m}^{-1}\). Equation 18) defines \(G\) experimentally as \(\frac{1}{\lambda_m}\). See Table 1. This definition makes the thermodynamic temperature identical to the colour temperature [1] if the latter is defined as the reciprocal of the wavelength of maximum emitted intensity.

**Graphical study**

Figure 1 is a plot of the apportioning function \((D_\lambda)\) for Lummer-Pringsheim [5,7] temperature values. These values were used in an 1899 study and are often used for reference. Figure 2 is a plot of the black body radiation law \((I_\lambda)\) for the same reference temperatures.

**Conclusion**

The analysis and graphical study thus show that the proposed radiation law (equation 16)) is in agreement with experimental observation. Furthermore it fulfils Einstein's desire for a solution of the black body radiation problem without the use of light quanta [8].
<table>
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<th>Temperature Scales</th>
<th>Fahrenheit</th>
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<th>Celsius</th>
<th>Kelvin</th>
<th>Wien-Gall</th>
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<td>°C</td>
<td>°K</td>
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<td>2.90 × 10⁻³ mK</td>
<td>1</td>
<td>1</td>
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</table>

**Table 1 : Reference Temperatures for the different Scales**
Figure 1: Gall Normalized Apportioning Function for Lummer-Pringsheim Temperatures
Figure 2: Gall Black Body Radiation Curves for Lummer-Pringsheim Temperatures
References


